Chapter 3

Vector Analysis

Contents

This chapter departs from the study and analysis of electromagnetic concepts where 1D scalar quantities was sufficient. Voltage, current, time, and 1D position will continue to be quantities of interest, but more is needed to prepare for future chapters.

In what lies ahead the vector field quantities **E** and **H** are of central importance. To move forward with this agenda we will start with a review of *vector algebra*, review of some analytic geometry, review the *orthogonal coordinate systems* Cartesian (rectangular), cylindrical, and spherical, then enter into a review of vector calculus. The depth of this last topic will likely be more intense than any earlier experiences you can remember.

3.1 Basic Laws of Vector Algebra

- The Cartesian coordinate system should be familiar to you from earlier math and physics courses
- The vector **A** is readily written in terms of the cartesian unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

- In linear algebra $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are known as basis vectors, each having unit length, i.e., $|\hat{\mathbf{x}}|$ and mutually orthogonal
- Also, the length of **A** is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

  and the unit vector in the **A** direction is

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

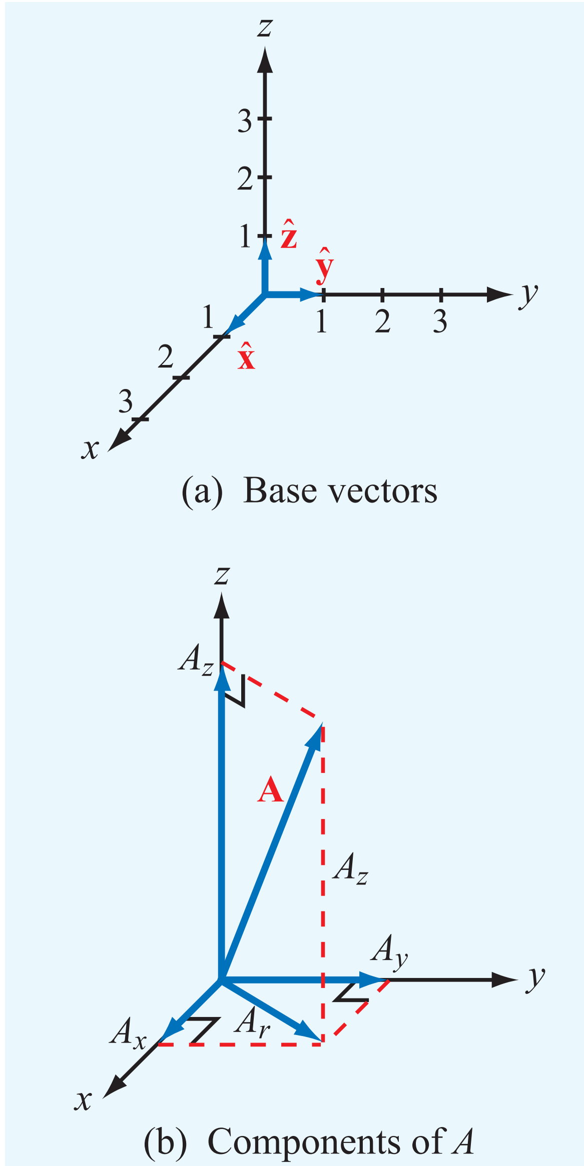


Figure 3.1: Expressing the vector **A** in terms the Cartesian unit vectors.

3.1.1 Equality of Two Vectors

- Vectors **A** and **B** are equal if their components are equal, i.e., $A_x = B_x$, etc.

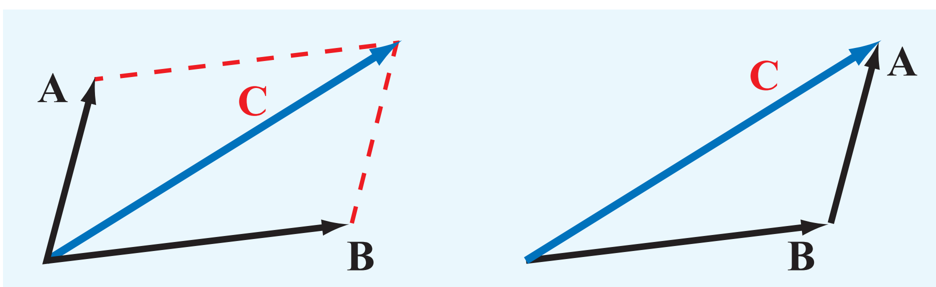
3.1.2 Vector Addition and Subtraction

- Addition of vectors means that the individual components are added together, that is

$$\begin{aligned} \mathbf{C} &= \mathbf{A} + \mathbf{B} \\ &= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z), \end{aligned}$$

thus $C_x = A_x + B_x$, etc.

- Visually you can utilize the *head-to-tail* or *parallelogram* rules



(a) Parallelogram rule (b) Head-to-tail rule

Figure 3.2: Vector addition rules.

- Vector subtraction is similar

$$\begin{aligned} \mathbf{D} &= \mathbf{A} - \mathbf{B} \\ &= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z), \end{aligned}$$

thus $D_x = A_x - B_x$, etc.

3.1.3 Position and Distance Vectors

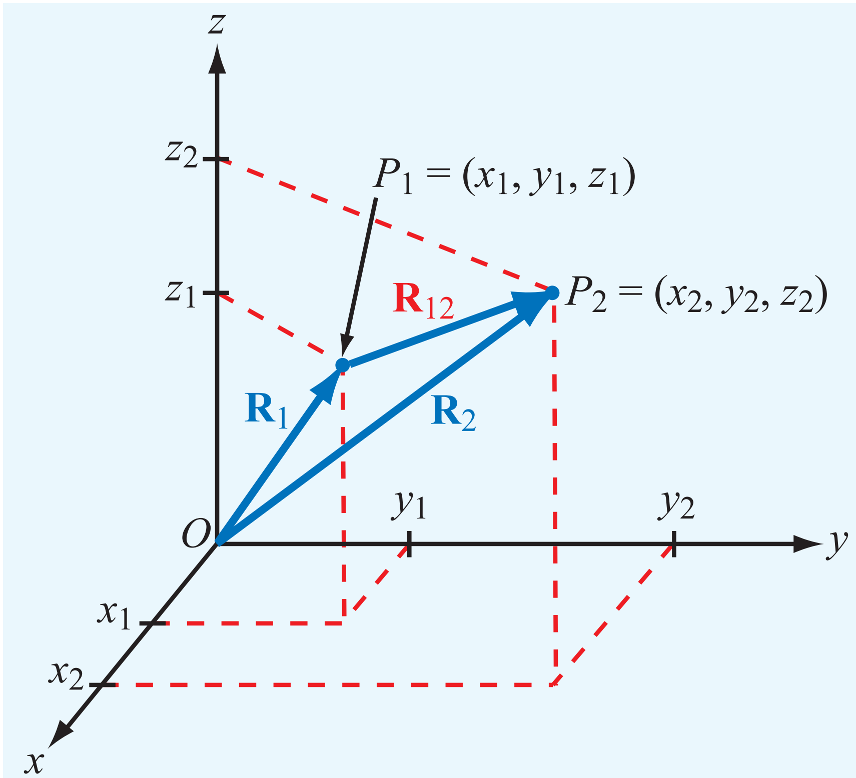


Figure 3.3: The notion of the *position* vector to a point, P_i, $\mathbf{R}_i$, and distance between, P_i and P_j, $\mathbf{R}_{ij}$ are vectors.

- Formally a position vector starts at the origin, so we use the notation

$$\mathbf{R}_i = \overrightarrow{OP_i} = \hat{\mathbf{x}}x_i + \hat{\mathbf{y}}y_i + \hat{\mathbf{z}}z_i$$

  where x_i, y_i, and z_i correspond to the point $P_i = (x_i, y_i, z_i)$

- The scara distance between two points is just $d = |\mathbf{R}_{ij}|$

$$d = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

3.1.4 Vector Multiplication

- Vector multiplication takes the form
  - *scalar* $\times$ *vector*:
  $$\boxed{\mathbf{B} = k\mathbf{A} = \text{element-by-element multiply by } k}$$
  - *scalar product* or *dot product*:
  $$\boxed{\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}}$$
  where θ_{AB} is the angle between the vectors (as in linear algebra)
  - **Note**: $A\cos\theta_{AB}$ is the component of $\mathbf{A}$ along $\mathbf{B}$ and $B\cos\theta_{AB}$ is the component of $\mathbf{B}$ along $\mathbf{A}$
  - Also,
  $$\begin{aligned} \mathbf{A} \cdot \mathbf{A} &= |\mathbf{A}|^2 = A^2 \\ A &= |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \end{aligned}$$
  - Using the inverse cosine
  $$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt{\mathbf{A} \cdot \mathbf{A}}\sqrt{\mathbf{B} \cdot \mathbf{B}}}\right]$$
  - Finally,
  $$\mathbf{A} \cdot \mathbf{A} = A_x B_x + A_y B_y + A_z B_z$$
  - Commutative and Distributive
  $$\boxed{\begin{gathered} \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \end{gathered}}$$

- *Vector product* or *cross product*:

$$\boxed{\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the plane containing $\mathbf{A}$ and $\mathbf{B}$ (see picture below for details)

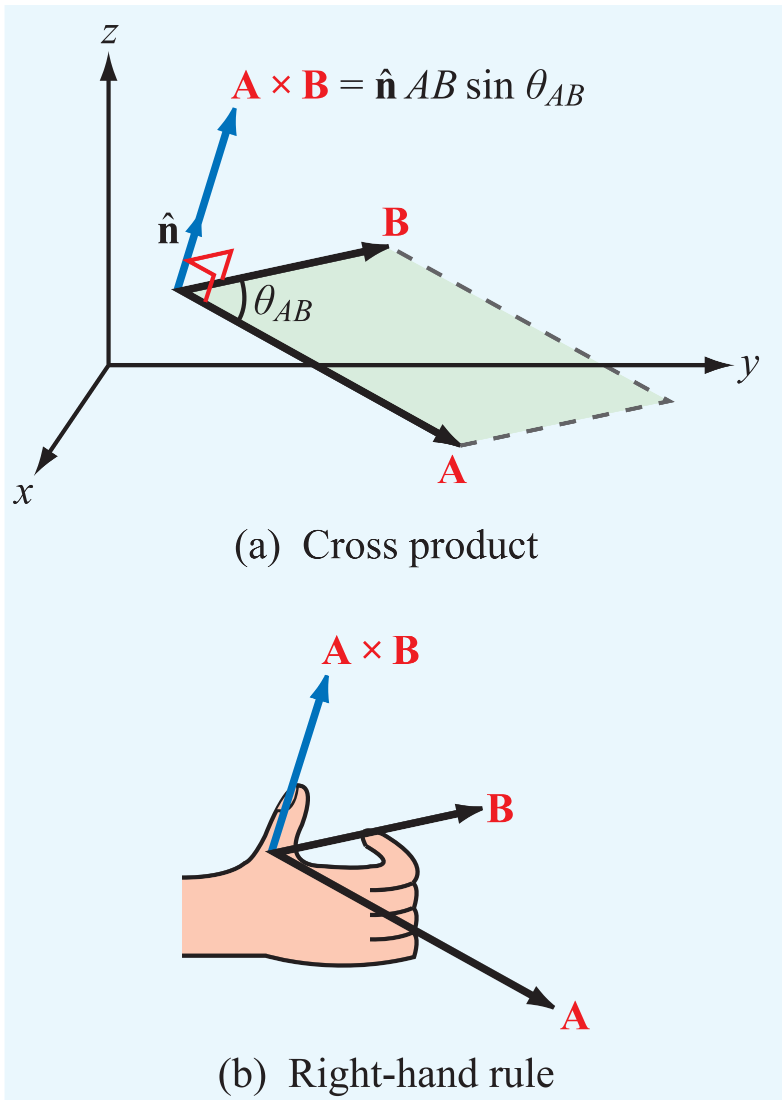


Figure 3.4: The cross product $\mathbf{A} \times \mathbf{B}$ and the *right-hand rule*.

  – The cross product is *anticommuntative*

$$\boxed{\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}}$$

- The cross product is *distributive*

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

- To calculate use the determinant formula

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) \\ &\quad + \hat{\mathbf{z}}(A_x B_y - A_y B_x) \end{aligned}$$

3.1.5 Scalar and Vector Triple Products

- Certain, *make sense*, vector products arise in electromagnetics

Scalar Triple Product

- Definition:

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \\ &= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \end{aligned}$$

Vector Triple Product

- Definition

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

- **Note**:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

- It can however be shown that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}),$$

which is known as the “bac-cab” rule

Example 3.1: Numpy for Vector Numerics

- To make things more convenient define the helper function `vec_fmt` (see Chapter 3 Jupyter notebook)

Basic Operations with Vectors

```
P1 = array([2,3,3])
P2 = array([1,-2,2])
A = P1
B = P2 - P1
print('A = ' + vec_fmt(A))
print('|A| = %1.3f' % (norm(A),))
print('B = ' + vec_fmt(B))
print('|B| = %1.3f' % (norm(B),))
print('hat(a) = ' + vec_fmt(A/norm(A)))
print('A dot B = %1.3f' % (dot(A,B),))
print('A cross B = ' + vec_fmt(cross(A,B)))
```

```
A = [2.000 unit_x, 3.000 unit_y, 3.000 unit_z]
|A| = 4.690
B = [-1.000 unit_x, -5.000 unit_y, -1.000 unit_z]
|B| = 5.196
hat(a) = [0.426 unit_x, 0.640 unit_y, 0.640 unit_z]
A dot B = -20.000
A cross B = [12.000 unit_x, -1.000 unit_y, -7.000 unit_z]
```

Figure 3.5: Using Numpy for basic vector numerical calculations.

Some Angles

```
beta = arccos(dot(A,[0,1,0])/norm(A))*180/pi
print('beta = %1.2f (deg)' % beta)
theta_AB = arccos(dot(A,B)/(norm(A)*norm(B)))*180/pi
print('theta_AB = %1.2f (deg)' % theta_AB)
angle_vecB_zaxis = arccos(dot(B,[0,0,1])/norm(B))*180/pi
print('angle_vecB_zaxis = %1.2f (deg)' % angle_vecB_zaxis)
```

```
beta = 50.24 (deg)
theta_AB = 145.15 (deg)
angle_vecB_zaxis = 101.10 (deg)
```

Vector Products

```
A = array([1,-1,2])
B = array([0,1,1])
C = array([-2,0,3])
print('(A x B) x C = ' + vec_fmt(cross(cross(A,B),C)))
print('A x (B x C) = ' + vec_fmt(cross(A,cross(B,C))))
print('B(A dot C) - C(A dot B) = ' + vec_fmt(B*dot(A,C)-C*dot(A,B)))
```

```
(A x B) x C = [-3.000 unit_x, 7.000 unit_y, -2.000 unit_z]
A x (B x C) = [2.000 unit_x, 4.000 unit_y, 1.000 unit_z]
B(A dot C) - C(A dot B) = [2.000 unit_x, 4.000 unit_y, 1.000 unit_z]
```

Figure 3.6: Using Numpy for more vector numerical calculations.

Example 3.2: TI Nspire CAS

- The TI nspire CAS can do both numerical and symbolic calculations
- Numerical examples are given below

TI Nspire CAS: Portions of Text Example 3-1

$p1 := \begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$
$a := p1$	$\begin{bmatrix} 2 & 3 & 3 \end{bmatrix}$
$p2 := \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$
$b := p2 - p1$	$\begin{bmatrix} -1 & -5 & -1 \end{bmatrix}$
$\mathrm{unitV}(a)$	$\begin{bmatrix} \frac{\sqrt{22}}{11} & \frac{3\cdot\sqrt{22}}{22} & \frac{3\cdot\sqrt{22}}{22} \end{bmatrix}$
$\mathrm{dotP}(a,b)$	-20
$\mathrm{crossP}(a,b)$	$\begin{bmatrix} 12 & -1 & -7 \end{bmatrix}$
$\beta := \cos^{-1}\left(\frac{\mathrm{dotP}(a,\begin{bmatrix} 0 & 1 & 0 \end{bmatrix})}{\mathrm{norm}(a)}\right)\cdot\frac{180}{\pi}$	50.2378
$\theta ab := \cos^{-1}\left(\frac{\mathrm{dotP}(a,b)}{\mathrm{norm}(a)\cdot \mathrm{norm}(b)}\right)\cdot\frac{180}{\pi}$	145.146

Figure 3.7: Using the TI Nspire CAS for vector numerics.

3.2 Orthogonal Coordinate Systems

- There three orthogonal coordinate systems in common usage in electromagnetics:
  - The Cartesian or rectangular system: $\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$
  - The cylindrical system: $\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$
  - The spherical system: $\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$

Table 3.1: Vector relations in the three common coordinate systems.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

- The three systems are needed to best fit the problem geometry at hand

3.2.1 Cartesian Coordinates

- We will have need of differential quantities of length, area and volume

Differential Length

$$d\mathbf{l} = \hat{\mathbf{x}} d l_x + \hat{\mathbf{y}} d l_y + \hat{\mathbf{z}} d l_z = \hat{\mathbf{x}} dx + \hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz$$

Differential Area

- A vector, $d\mathbf{s}$, that is normal to the two coordinates describing the scalar area ds

- There are three different differential areas, $d\mathbf{s}$, to consider:

$$\begin{aligned} d\mathbf{s}_x &= \hat{\mathbf{x}}\, d l_y\, d l_z = \hat{\mathbf{x}}\, dy\, dz \quad (y - z\text{-plane}) \\ d\mathbf{s}_y &= \hat{\mathbf{x}}\, dx\, dz \quad (x - z\text{-plane}) \\ d\mathbf{s}_z &= \hat{\mathbf{x}}\, dx\, dy \quad (x - y\text{-plane}) \end{aligned}$$

Differential Volume

$$d\mathcal{V} = dx\, dy\, dz$$

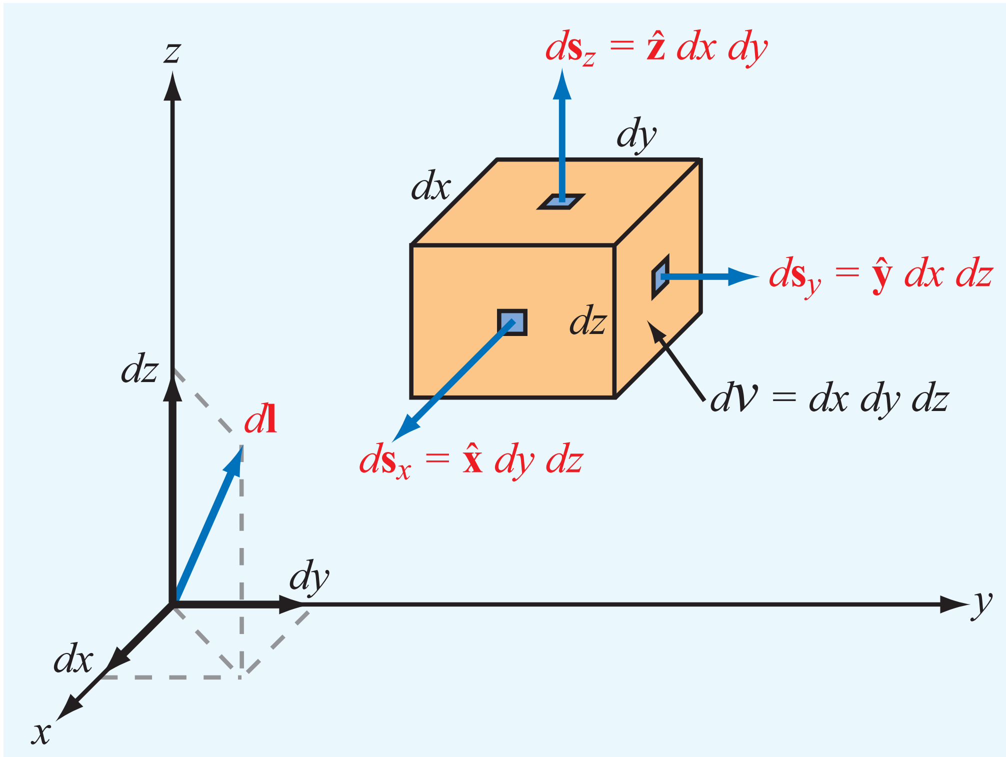


Figure 3.8: Differential length, area, and volume.

3.2.2 Cylindrical Coordinates

- The cylindrical system is used for problems involving cylindrical symmetry
- It is composed of: (1) the radial distance $r \in [0, \infty)$, (2) the azimuthal angle, $\phi \in [0, 2\pi)$, and $z \in (-\infty, \infty)$, which can be thought of as height
- As in the case of the Cartesian system, $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular or orthogonal to each other, e.g., $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$, etc.
- Likewise the cross product of the unit vectors produces the cyclical result

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$$

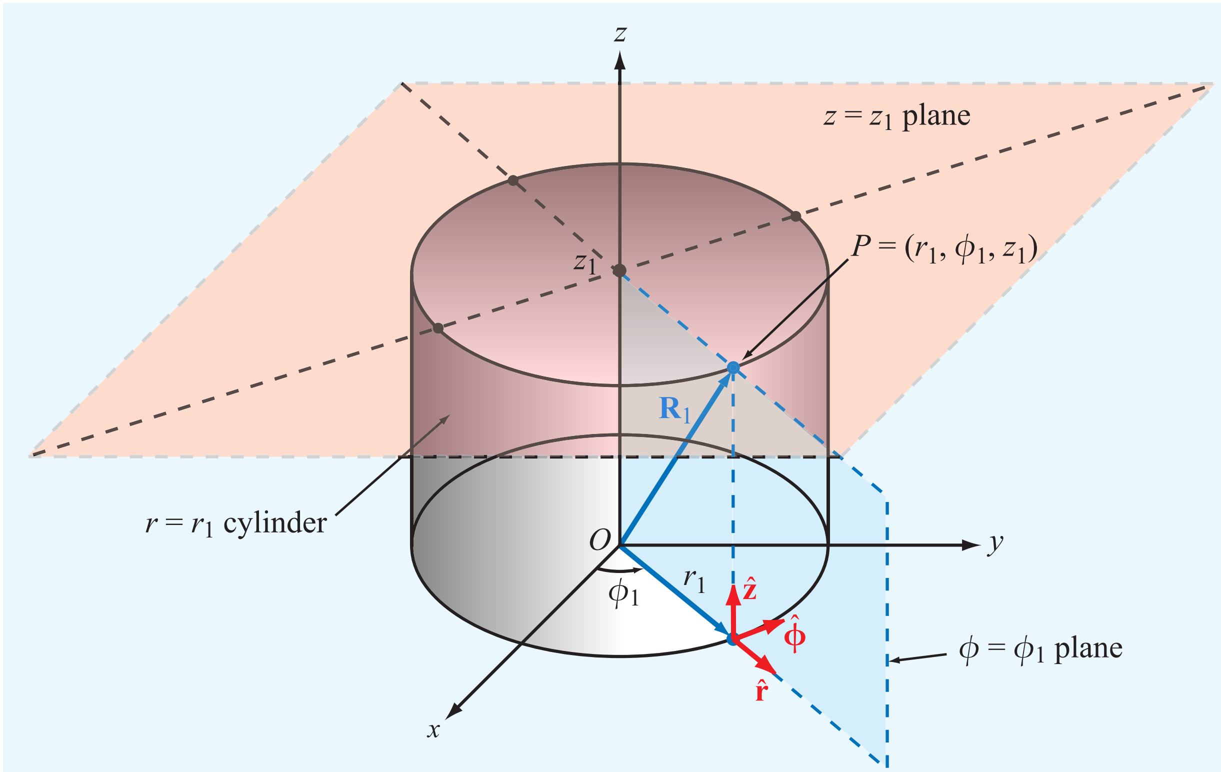


Figure 3.9: A point in the cylindrical system.

- The general vector expansion

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$$

  is obvious, as is the scalar length

$$|\mathbf{A}| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$$

- Looking at 3.9 it is interesting to note that $\hat{\boldsymbol{\phi}}$ is absent in the position vector

$$\overrightarrow{OP} = \hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1,$$

  but is present in the point $P(r_1, \phi_1, z_1)$ itself

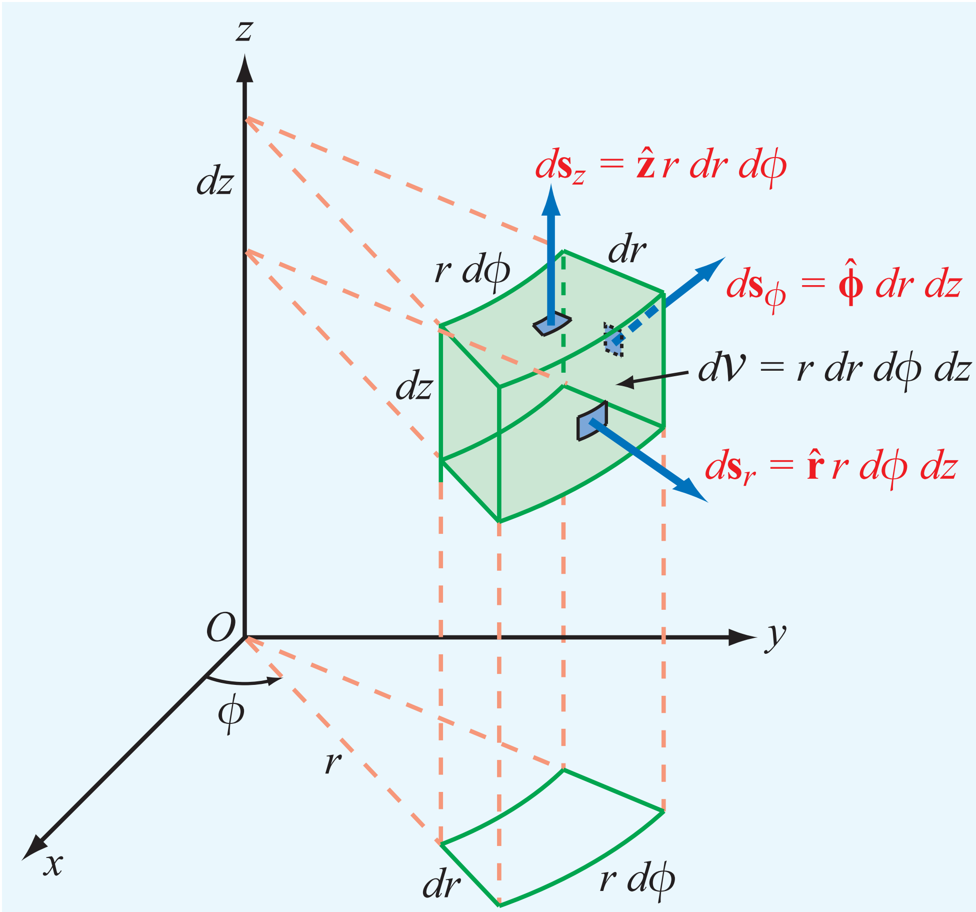


Figure 3.10: Differential quantities in the cylindical system.

Differential Quantities

- The differential quantities do not follow from the Cartesian system

- The differential length of the azimuthal component is also a function of the radial component, i.e.,

$$dl_r = dr, \quad dl_\phi = rd\phi, \quad dl_z = dz$$

- In the end

$$d\mathbf{l} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\phi}}rd\phi + \hat{\mathbf{z}}dz$$

- The differential surface follows likewise

$$
\begin{aligned}
d\mathbf{s}r &= \hat{\mathbf{r}}\, r\, d\phi\, dz \quad (\phi - z \text{ cylindrical surface}) \\
d\mathbf{s}_\phi &= \hat{\boldsymbol{\phi}}\, dr\, dz \quad (r - z \text{ plane}) \\
d\mathbf{s}_z &= \hat{\mathbf{z}}\, dr\, d\phi \quad (r - \phi \text{ plane})
\end{aligned}
$$

- The differential area is likely the most familiar from calculus

$$
d\mathcal{V} = r\, dr\, \phi\, dz
$$

Example 3.3: Distance Vector from z-Axis to $r - \phi$-Plane

- When making field calculations due to charge or current along a line, we need the distance vector shown below:

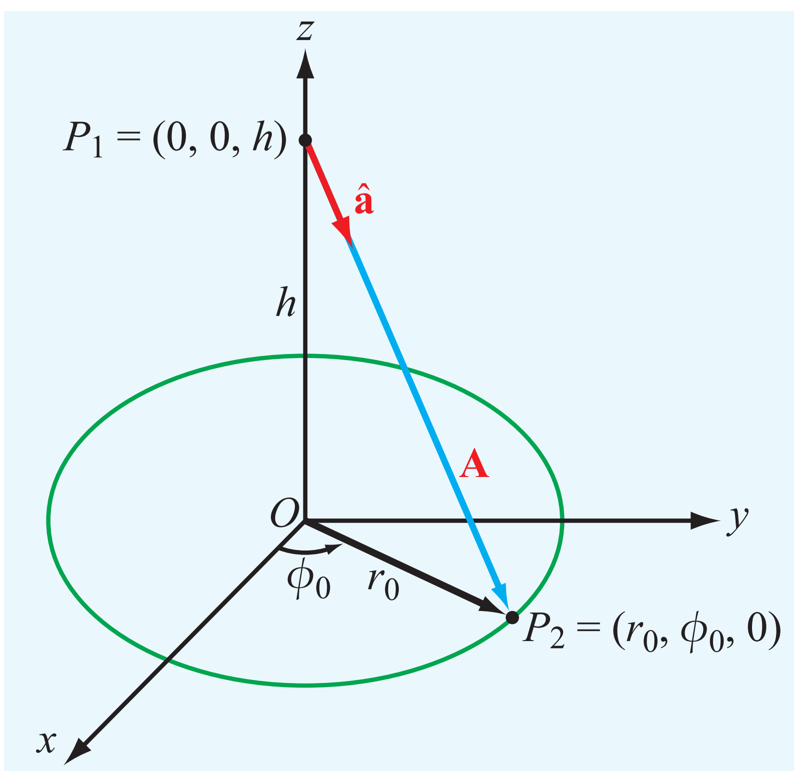


Figure 3.11: Distance vector from z-axis to point in $r - \phi$ plane.

- The vector from a point P_1 on the z-axis, $(0, 0, h)$, to a point P_2 in the $r\phi$-plane, $(r_0, \phi_0, 0)$, is

$$\mathbf{A} = \overrightarrow{OP}_2 - \overrightarrow{OP}_1 = \hat{\mathbf{r}}r_0 - \hat{\mathbf{z}}h$$

- The unit vector is

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{r}}r_0 - \hat{\mathbf{z}}h}{\sqrt{r_0^2 + h^2}}$$

Note: ϕ is not present!

- Once $\phi = \phi_0$ is specified the unambiguous point direction resolved

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Example 3.4: **Volume of a Cylinder**

- Consider a cylinder of height 2 cm and diameter 3 cm
- Using simple calculus, the surface area of the cylinder is

$$S = \int_0^2 \int_0^{2\pi} r \, d\phi \, dz \bigg|_{r=3/2} = 6\pi \quad (\text{cm})^2$$

- The volume of the cylinder is

$$\mathcal{V} = \int_0^{3/2} \int_0^2 \int_0^{2\pi} r \, d\phi \, dz \, dr = 4\pi \cdot \frac{r^2}{2}\bigg|_0^{3/2} = \frac{9\pi}{2} \quad (\text{cm})^3$$

3.2.3 Spherical Coordinates

- In this coordinate system a single range variable R plus two angle variables θ and ϕ are employed

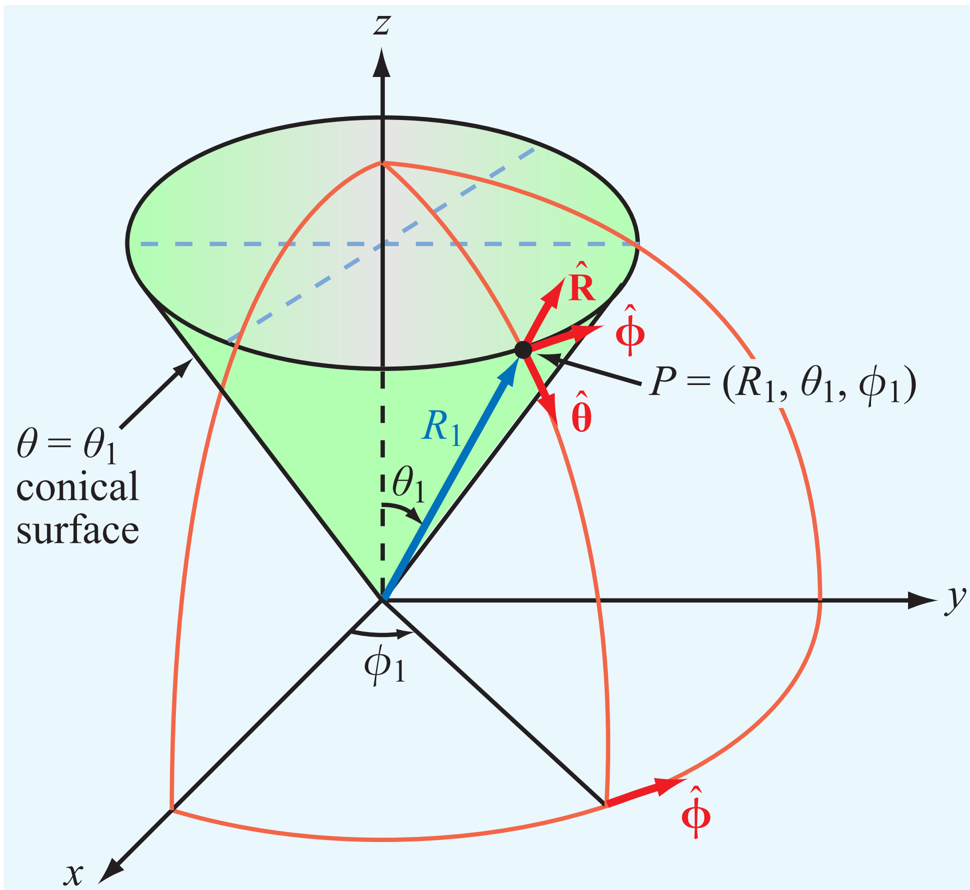

Figure 3.12: The spherical coordinate system showing a point P_1 and position vector $\hat{\mathbf{R}}_1$.

- It is composed of: (1) the radial distance $r \in [0, \infty)$, (2) the azimuthal angle (same as cylindrical), $\phi \in [0, 2\pi)$, and the *zenith angle* $\theta \in [0, \pi]$, which is measured from the positive z-axis

- All coordinates are again mutually orthogonal to span a 3D space

- The cross product of the unit vectors produces the cyclical result

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$$

- The general vector expansion

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$$

  is obvious, as is the scalar length

$$|\mathbf{A}| = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$$

- The position vector $\mathbf{R}_1$ (3.12) is

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{R}}R_1,$$

  but needs knowledge of θ_1 and ϕ_1 to be complete

Differential Quantities

- The differential quantities are different yet again from the Catestian and the cylindrical systems

- The differential length of the zenith component is like the azimuthal component in the cylindrical system

- The differential length of the azimuthal component is now a function of both the radial component and the zenith component, i.e.,

$$dl_R = dR, \quad dl_\theta = Rd\theta, \quad dl_\phi = R\sin\theta d\phi$$

- In the end

$$d\mathbf{l} = \hat{\mathbf{R}} dr + \hat{\boldsymbol{\theta}}\, R d\theta + \hat{\boldsymbol{\phi}}\, R \sin\theta dz$$

- The differential surface follows

$$\begin{aligned} d\mathbf{s}_R &= \hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi \quad (\theta - \phi \text{ spherical surface}) \\ d\mathbf{s}_\theta &= \hat{\boldsymbol{\theta}}\; R \sin\theta dR\, d\phi \quad (R - \phi \text{ conical plane}) \\ d\mathbf{s}_\phi &= \hat{\boldsymbol{\phi}} R\, dR\, d\theta \quad (R - \theta \text{ plane}) \end{aligned}$$

- Again the differential area is likely the most familiar from calculus

$$d\mathcal{V} = R^2 \sin\theta\, dR\, d\theta\, d\phi$$

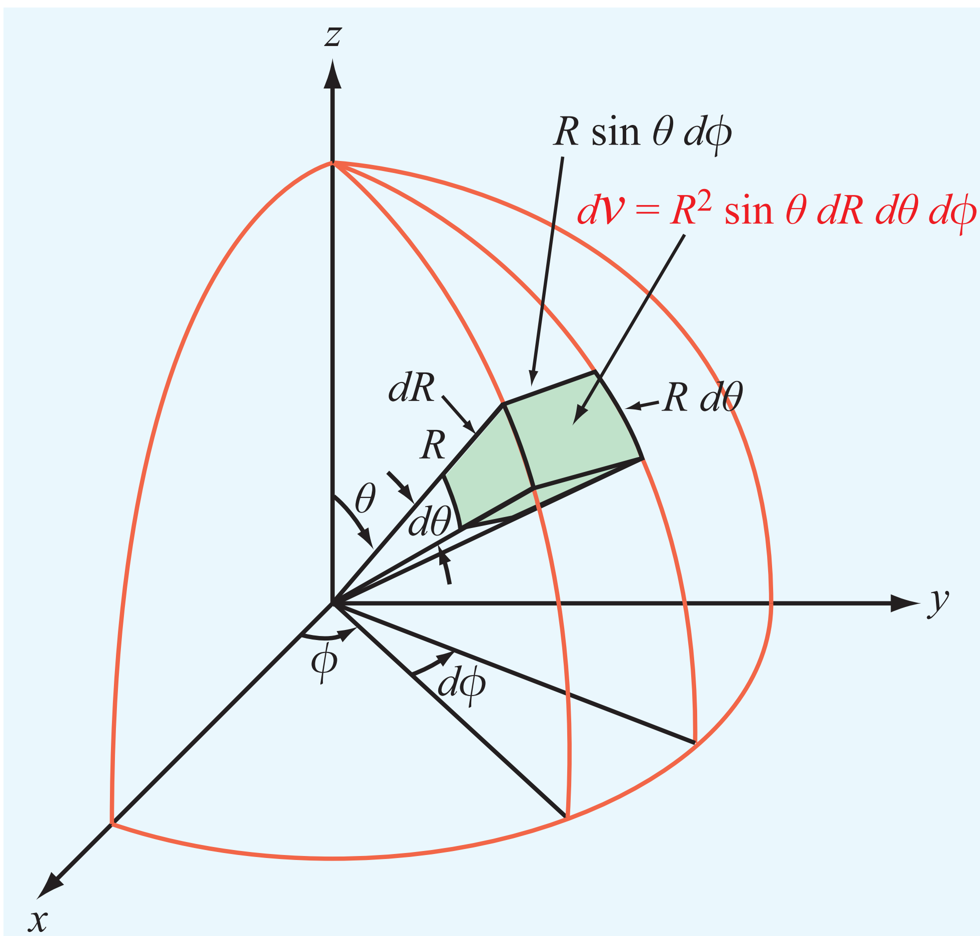

Figure 3.13: The spherical coordinate differential volume.

Example 3.5: **Preview of Chapter 4 - A Charge density**

- A volume charge density

$$\rho_v = 4\cos^2\theta \quad (\text{C/m}^3)$$

is present in a sphere of radius 2 cm

- To find the total charge in the sphere we integrate the charge density over the volume

$$\begin{aligned}
Q &= \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{0.02} \big(4\cos^2\theta\big) R^2 \sin\theta \, dR \, d\theta \, d\phi \\
&= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right)\Bigg|_0^{0.02} \sin\theta \cos^2\theta \, d\theta \, d\phi \\
&= \frac{32}{3} \times 10^{-6} \int_0^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi} d\phi \\
&= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi = \frac{128\pi}{9} \times 10^{-6} \\
&= 44.68 \quad (\mu\text{C})
\end{aligned}$$

Just a little calculus review, especially the anti-derivative of $\sin\theta\cos^2\theta$

3.3 Coordinate Transformations

- Overview of the various transformations: $(x, y, x) \Leftrightarrow (r, \phi, z)$, $(x, y, z) \Leftrightarrow (R, \theta, \phi)$, and $(r, \phi, z) \Leftrightarrow (R, \theta, \phi)$

Table 3.2: Coordinate transformations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

- There are three aspects of each *to and from* coordinate transformations:

  1. The coordinate variables – (x, y, z), (r, ϕ, z), and (R, θ, ϕ)
  2. The unit vectors – $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, $(\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}})$, and $(\hat{\mathbf{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$
  3. The vector components – (A_x, A_y, A_z), (A_r, A_ϕ, A_z), and (A_R, A_θ, A_ϕ)

3.3.1 Cartesian to Cylindrical Transformations

- This is the most obvious and most familiar

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \text{ (watch the quadrant)} \\
x &= r\cos\phi, \qquad y = r\sin\phi \\
z &= z
\end{aligned}
$$

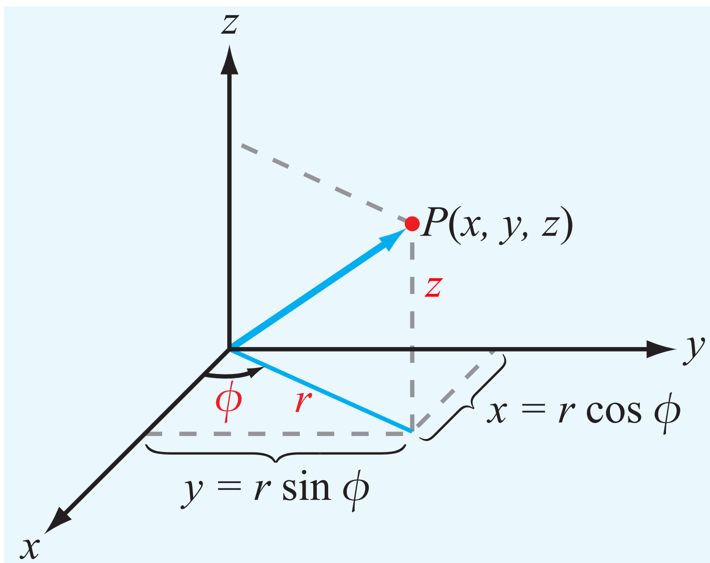

Figure 3.14: Cartesian and cylindrical variable relationships.

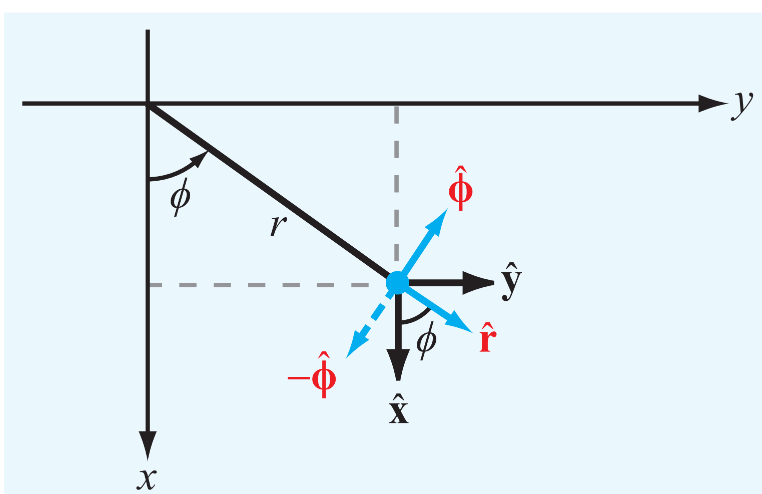

Figure 3.15: Cartesian and cylindrical unit vector relationships.

3.3.2 Cartesian to Spherical Transformations

- These are less familiar, but very useful in this course

$$R = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{(watch the quadrants)}$$

$$x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi$$

$$z = R\cos\theta$$

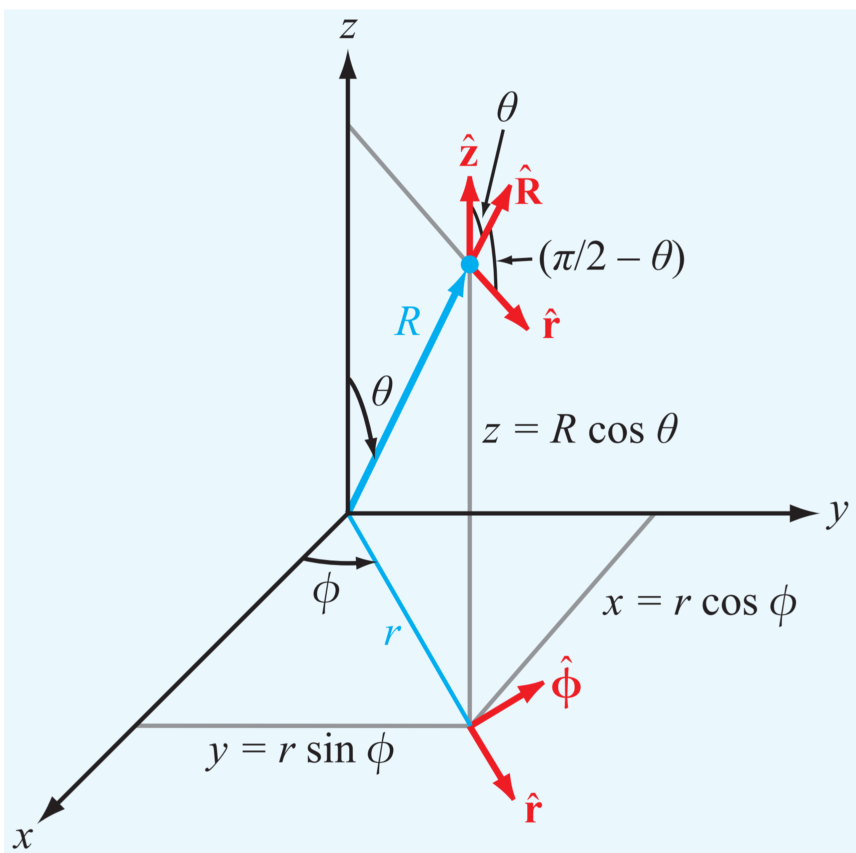

Figure 3.16: Cartesian and spherical variable and unit vector relationships.

3.3.3 Cylindrical to Spherical Transformations

- See Table 3.2

$$
\begin{aligned}
R &= \sqrt{r^2 + z^2}, \qquad \theta = \tan^{-1}(r/z), \qquad \phi = \phi \\
r &= R\sin\theta, \qquad \phi = \phi, \qquad z = R\sin\theta
\end{aligned}
$$

3.3.4 Distance Between Two Points

- The distance between two points, $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, arises frequently

- In **Cartesian** coordinates the answer is obvious

$$d = |\mathbf{R}_{12}| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

- For the case of **cylindrical** coordinates we apply the variable transformations to arrive at

$$
\begin{aligned}
d &= \big[(r_2\cos\phi_2 - r_1\cos\phi_1)^2 + (r_2\sin\phi_2 - r_1\sin\phi_1)^2 \\
&\qquad + (z_2 - z_1)^2\big]^{1/2} \\
&= \left[r_2^2 + r_1^2 - 2r_1r_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}
\end{aligned}
$$

- Finally, for the **spherical** coordinates

$$
\begin{aligned}
d &= \big\{R_2^2 + R_1^2 - 2R_1R_2[\cos\theta_2\cos\theta_1 \\
&\qquad + \sin\theta_2\sin\theta_1\cos(\phi_2 - \phi_1)]\big\}^{1/2}
\end{aligned}
$$

3.4 Gradient of a Scalar Field

- In this section we deal with the rate of change of a scalar quantity with respect to position in all three coordinates (x, y, z)
- The result will be a vector quantity as the maximum rate of change of the scalar quantity will have direction
  - Think of skiing down a mountain; if you want to descend as quickly as possible you ski the path the follows the negative of the maximum rate of change in elevation
  - The route corresponds to the negative of the *gradient*
- Suppose T represent the scalar variable of temperature in a material as a function of (x, y, z)
- The gradient of temperature T is written as

$$\boxed{\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}}$$

- **Note**: A differential change in the distance vector $d\mathbf{l}$ dotted with the gradient gives the scalar change in temperature, dT, i.e.

$$\begin{aligned} dT &= \nabla T \cdot d\mathbf{l} \\ &= \nabla T \cdot \big(\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz\big) \\ &= \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz \end{aligned}$$

- As an operator we can write the so-called *del operator* in **Cartesian** coordinates as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Directional Derivative

- In calculus you learn about the directional derivative

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l$$

as the derivative of T along $\hat{\mathbf{a}}$, which is the unit vector of the differential distance $d\,\mathbf{dl} = \hat{\mathbf{a}}_l dl$

- A nice extension is to find the difference $T_2 - T_1$, which corresponds to points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$

- We integrate both side of the directional derivative definition to obtain

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}$$

Example 3.6: Directional Derivative of $T = x^2 + y^2 z$

- We seek the directional derivative of T along the direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ evaluated at $(1, -1, 2)$

- Start by finding the gradient

$$\nabla T = \hat{\mathbf{x}}\,2x + \hat{\mathbf{y}}\,2yz + \hat{\mathbf{z}}\,y^2$$

- Note that
$$\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2,$$
so
$$\hat{\mathbf{a}}_l = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}$$

- The directional derivative is
$$\begin{aligned}\frac{dT}{dl} &= \left(\hat{\mathbf{x}}\,2x + \hat{\mathbf{y}}\,2yz + \hat{\mathbf{z}}\,y^2\right) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}\right)\\ &= \frac{4x + 6yz - 2y^2}{\sqrt{17}},\end{aligned}$$

- At the point $(1, -1, 2)$ we finally have
$$\left.\frac{dT}{dl}\right|_{(1,-1,2)} = \frac{-10}{\sqrt{17}} = -0.588$$

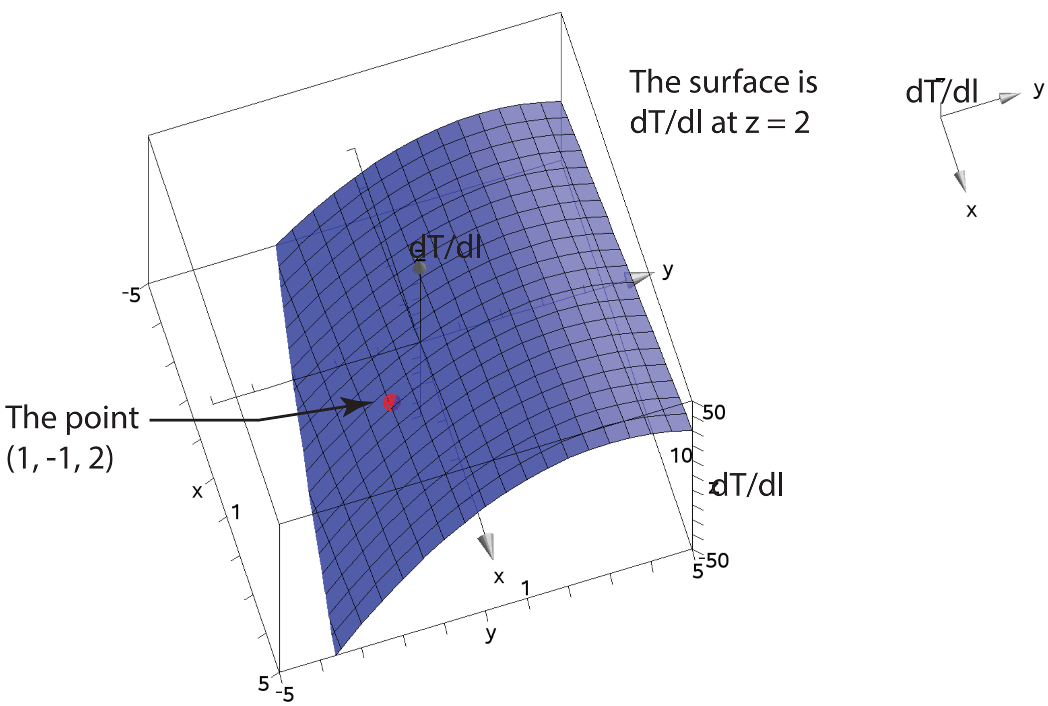

Figure 3.17: The directional derivative, dT/dl, as a surface over (x, y) with z fixed at 2.

3.4.1 Gradient Operator in Cylindrical and Spherical Coordinates

- To move forward with the expressing gradient in the other two coordinate systems, requires a bit of calculus
- For **cylindrical** coordinates it can be shown that

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

- For **spherical** coordinates it can be shown that

$$\nabla = \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$$

3.4.2 Properties of the Gradient Operator

- From basic calculus it follows that

$$\begin{aligned} \nabla\big(U + V\big) &= \nabla U + \nabla V \\ \nabla\big(UV\big) &= U\,\nabla U + V\,\nabla U \\ \nabla V^n &= nV^{n-1}\,\nabla V, \text{ for any } n \end{aligned}$$

Example 3.7: **Gradiant of** V

- Consider the scalar function

$$V = x^2y + xy^2 + xz^2$$

- The gradient is simply

$$\nabla V = \hat{\mathbf{x}}(2xy + y^2 + z^2) + \hat{\mathbf{y}}(x^2 + 2xy) + \hat{\mathbf{z}}(2xz)$$

- At the point $P_1 = (1, -1, 2)$ the gradiant vector is

$$\nabla V(1, -1, 2) = \hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}4$$

3.5 Divergence of a Vector Field

- The divergence of a vector field is in a sense complementary to the gradient:

$$\begin{aligned} \text{Gradient of a scalar function} &\Rightarrow \text{Vector function} \\ \text{Divergence of a vector function} &\Rightarrow \text{Scalar function} \end{aligned}$$

- So what is it? Take a look at https://en.wikipedia.org/wiki/Divergence

- For a 3D *vector field* it measures the extent to which the vector field behaves as a *source* or *sink*

- A 3D field has *field lines* and corresponding *flux density*, which defines the outward flux crossing a unit surface ds

  - **For the EE**: Consider a point charge $+q$; if we place a sphere (infinitesimally small) around it, there will be a net flow of flux over the surface of the sphere; move the sphere away from the charge location and the net flow of flux (in/out) is zero

  - **For the ME**: Consider heating or cooling of air in a region; the *velocity* of the air, which is influenced by the heating, is a vector field; the velocity points outward from the heated region just like the electric field from the $+q$ charge

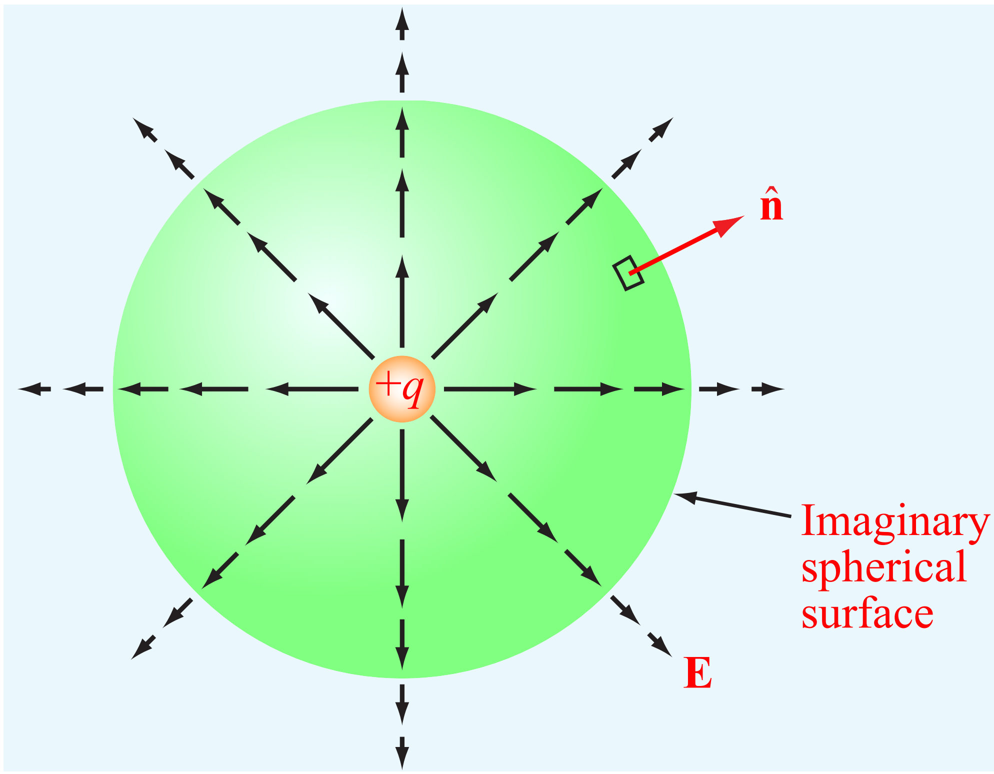

Figure 3.18: The electric field flux lines due to a point charge $+q$ are normal to a sphere ($\hat{\mathbf{n}}$) centered on the charge.

- In more detail the $+q$ charge produces flux density (outward flux crossing a unit surface)

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot d\mathbf{s}}{|d\mathbf{s}|} = \mathbf{E} \cdot \hat{\mathbf{n}}$$

where $d\mathbf{s}$ includes the orientation of the surface via $\mathbf{s}$ and the dot product insures that only the flux normal to the surface is accounted for; $\hat{\mathbf{n}}$ is the outward normal to the surface, i.e., $d\mathbf{s}/|d\mathbf{s}|$

- The total flux crossing a closed surface S (e.g., a sphere) is

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\hat{\mathbf{s}}$$

- For a general vector field, say $\mathbf{E}(x, y, z)\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z$, we can sum the outward flux through each of the faces of a differential cube as shown in Figure3.18

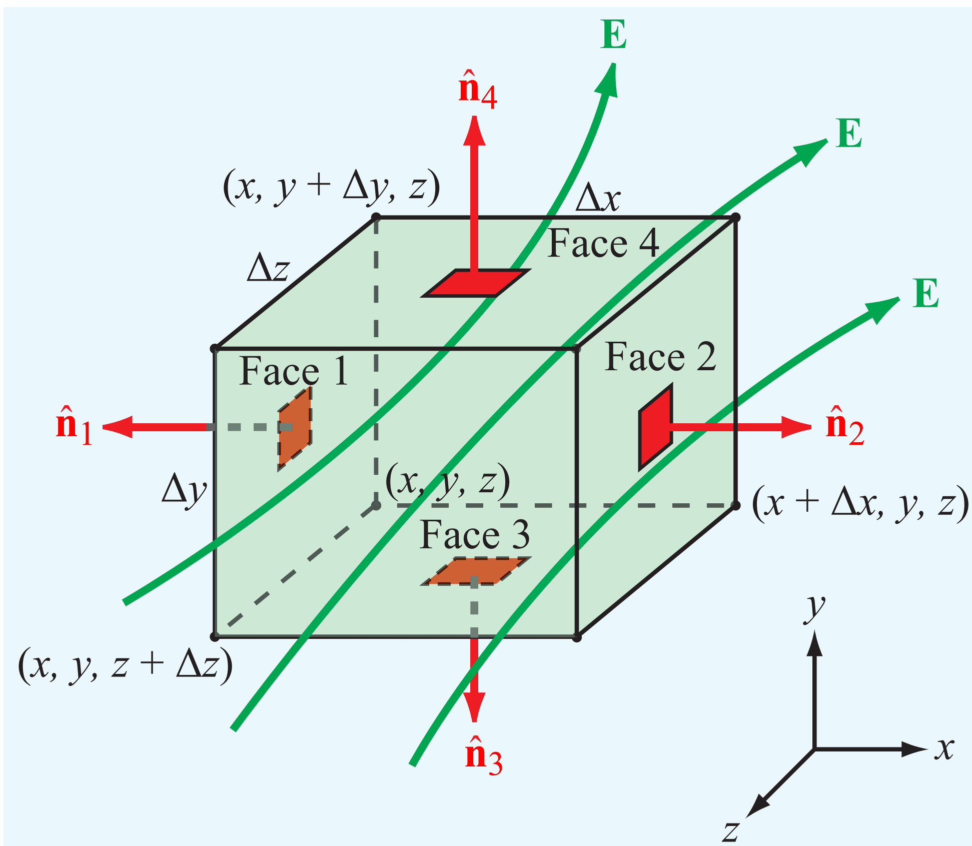

Figure 3.19: Detailing divergence by considering the flux exiting the six faces of a differential cube (parallelpiped).

- In the end we have

$$\oint_S \mathbf{E} \cdot d\,\mathbf{s} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = (\text{div}\mathbf{E})\ \Delta\mathcal{V}$$

- Now, we take the limit as $\Delta\mathcal{V} \to 0$ to obtain the formal definition of divergence

$$\nabla \cdot \mathbf{E} = \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

- If $\nabla \cdot \mathbf{E} > 0$ a *source* if present, while $\nabla \cdot \mathbf{E} < 0$ means a *sink* is present, and $\nabla \cdot \mathbf{E} = 0$ means the field is *divergenceless*

Divergence Theorem

- Moving forward into Chapter 4 we will quickly bump into the *divergence theorem*, which states that

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E} \, d\mathcal{V} = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Example 3.8: **Divergence in Cartesian Coordinates**

- Consider $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$ at the point $P_1 = (2, -2, 0)$
- Using the definition in Cartesian coordinates

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\partial 3x^2}{\partial x} + \frac{\partial 2z}{\partial y} + \frac{\partial x^2 z}{\partial z} \\ &= 6x + 0 + x^2 = x^2 + 6x\end{aligned}$$

- Evaluating at $(2, -2, 0)$ we have

$$\nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16$$

- The positive diverge at $(2, -2, 0)$ can be seen in a 3D vector slice plot from Mathematica

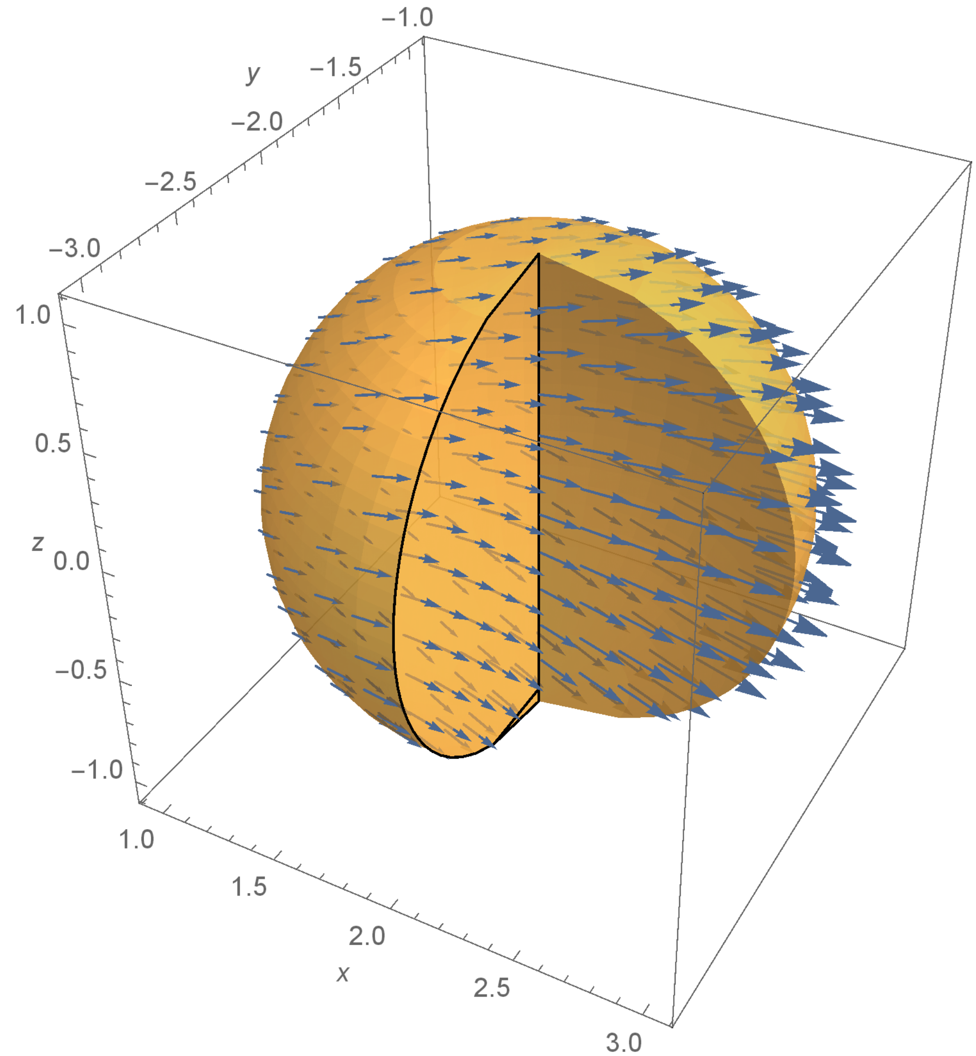

Figure 3.20: 3D vector field plot from Mathematica with a cut-sphere centered at $(2, -2, 0)$; the positive divergence is clear.

Example 3.9: Divergence in Spherical Coordinates

- Working a diverge calculation in cylindical or spherical requires the formulas inside the back cover of the text
- For the problem at hand we have
$$\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta/R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta/R^2),$$
which is in spherical coordinates
- Find the divergence at $P_2 = (a/2, 0, \pi)$
$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{1}{R^2}\frac{\partial}{\partial R}\big(R^2 E_R\big) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\big(E_\theta \sin\theta\big)\\ &\quad + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi}\\ &= \frac{1}{R^2}\frac{\partial}{\partial R}\big(a^3\cos\theta\big) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right)\\ &= -\frac{2a^3\cos\theta}{R^3}\end{aligned}$$
- At the point $(a/2, 0, \pi)$ we have
$$\nabla\cdot\mathbf{E}\Big|_{(a/2,0,\pi)} = -16$$
- Since the divergence is negative at this point, we conclude that a field sink is present
- A 2D vector plot (Python, Mathematica, or MATLAB) can be used to review the field behavior using arrows

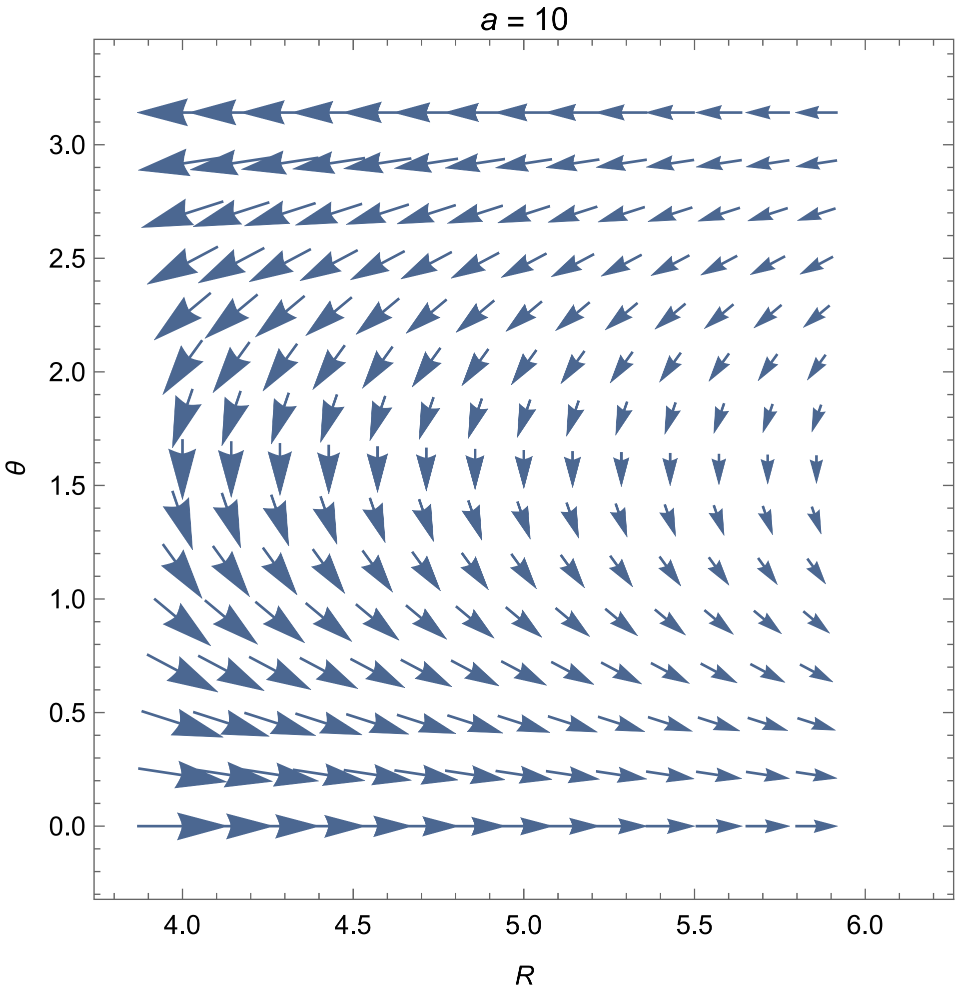

Figure 3.21: 2D vector field plot for $a = 10$ in just the R and θ axes making the negative divergence at $(5, 0, \pi)$ clear.

Example 3.10: Ulaby 3.44b

- Each of the following vector fields is displayed below in the form of a vector representation. Determine $\nabla \cdot \mathbf{A}$ analytically and then compare the results with your expectations on the basis of the displayed pattern.

- Worked using the Jupyter notebook (screen shots)

Part b

$\mathbf{A} = -\hat{\mathbf{x}}\ \sin 2y + \hat{\mathbf{y}}\ \cos 2x$, for $-\pi \leq x, y \leq \pi$.

- The divergence calculation in Cartesian coordinates is just

$$\nabla \cdot \mathbf{E} = \frac{\partial}{\partial x}\big(- \sin 2y\big) + \frac{\partial}{\partial y}\big(\cos 2x\big) = 0$$

- The fact that $\nabla \cdot \mathbf{E} = 0$ means that the arrow/quiver plot should show everywhere that the net input and output of flux over \textit{any} 2D region is zero
- The quiver plot is reproduced below using Python (Mathematica or MATLAB also work):

```
# Plot for Part a
X, Y = meshgrid(arange(-pi, pi+.5, .5), arange(-pi, pi+.5, .5))
U = -sin(2*Y)
V = cos(2*X)
figure(figsize=(5,5))
Q = quiver(X, Y, U, V,pivot='mid',color='r')
title('2D Vector Field: $\mathbf{A} = -\hat{\mathbf{x}}\,-\sin 2y \
       + \hat{\mathbf{y}}\, \cos 2x$')
# plot a couple circles to help visualize in/out flux
t  = arange(0,2*pi,2*pi/200)
plot(cos(t)/4-1.5,sin(t)/4,'b')
plot(cos(t)/4+1.75,sin(t)/4+3,'b')
xlabel(r'x')
ylabel(r'y');
```

Figure 3.22: Ulaby problem 3.44b set-up in the Jupyter notebook.

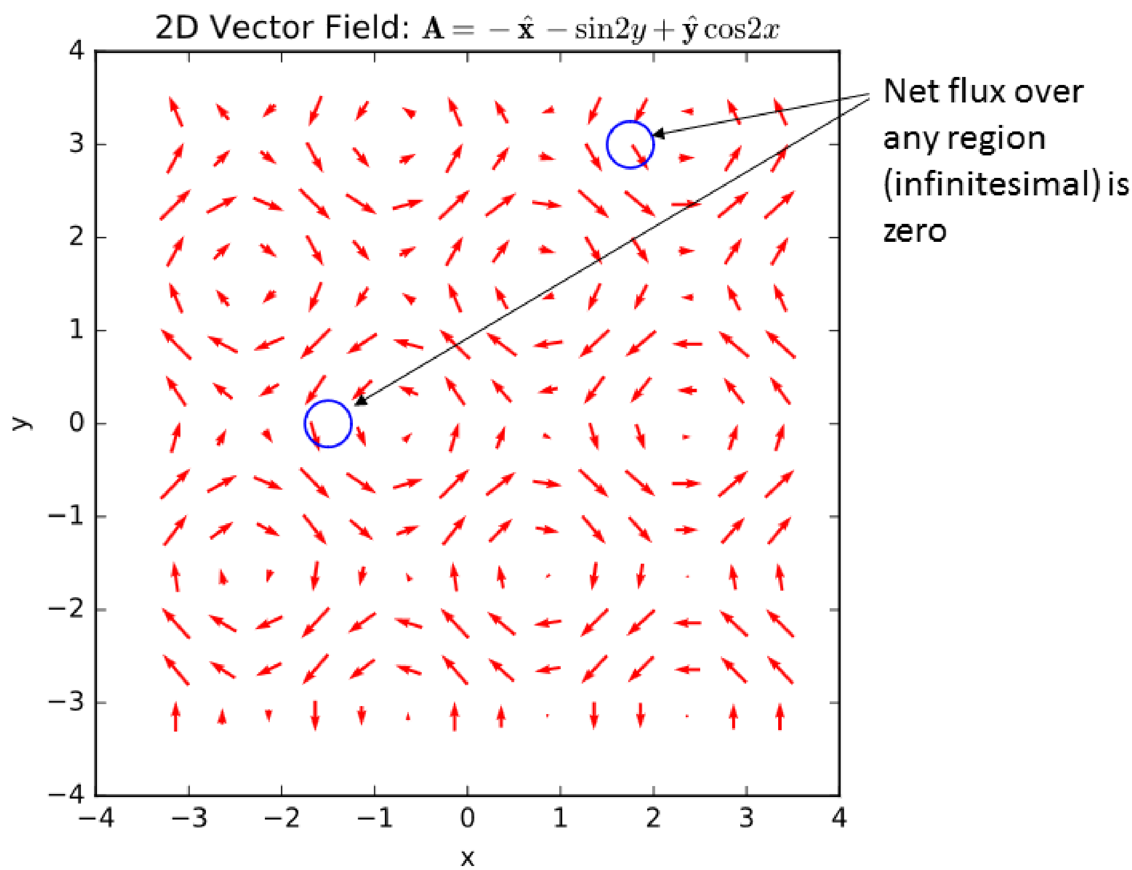

Figure 3.23: Ulaby problem 3.44b vector (quiver) plot in Jupyter notebook to verify divegence of zero.

3.6 Curl of a Vector Field

- Moving forward, the next vector operator, *Curl*, applies more often to magnetic fields; See https://en.wikipedia.org/wiki/Curl_(mathematics)

- The Curl describes the rotation of a 3D field, in an infinitesimal sense

- A field **B** has *circulation* if the line integral

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l} \neq 0$$

- For the case of a uniform field, e.g., $\mathbf{B} = \hat{\mathbf{x}} B_0$, forming a line integral around a closed rectangular contour in the $x - y$ plane yields zero, i.e.,

$$\begin{aligned}\text{Circulation} &= \int_a^b \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_b^c \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy \\ &\quad + \int_c^d \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{x}}\, dx + \int_d^a \hat{\mathbf{x}} B_0 \cdot \hat{\mathbf{y}}\, dy \\ &= B_0\, \Delta x - B_0\, \Delta x = 0\end{aligned}$$

- Futhermore, a small fictitious *paddle wheel* placed in the uniform field will not rotate, no matter the orientation of the wheel rotation axis

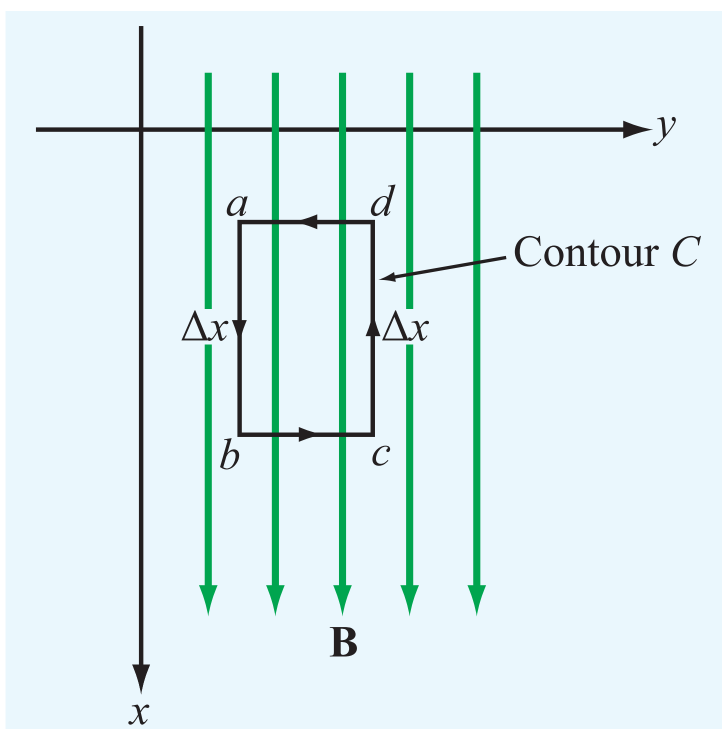

Figure 3.24: A uniform field, $\mathbf{B} = \hat{\mathbf{x}} B_0$ with circulation over C zero.

- Consider the azimuthal field of a wire carrying current I along the z-axis
- The magnetic flux in the$x - y$ plane follows $\hat{\boldsymbol{\phi}}$ with strength $\mu_0 I/(2\pi r)$
- To compute the circulation we consider differential length $d\mathbf{l} = \hat{\boldsymbol{\phi}} r\, d\phi$ and determine the circulation to be

$$\text{Circulation} = \int_0^{2\pi} \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} r\, d\phi = \mu_0 I$$

- Clearly a paddle wheel placed in this field will rotate!

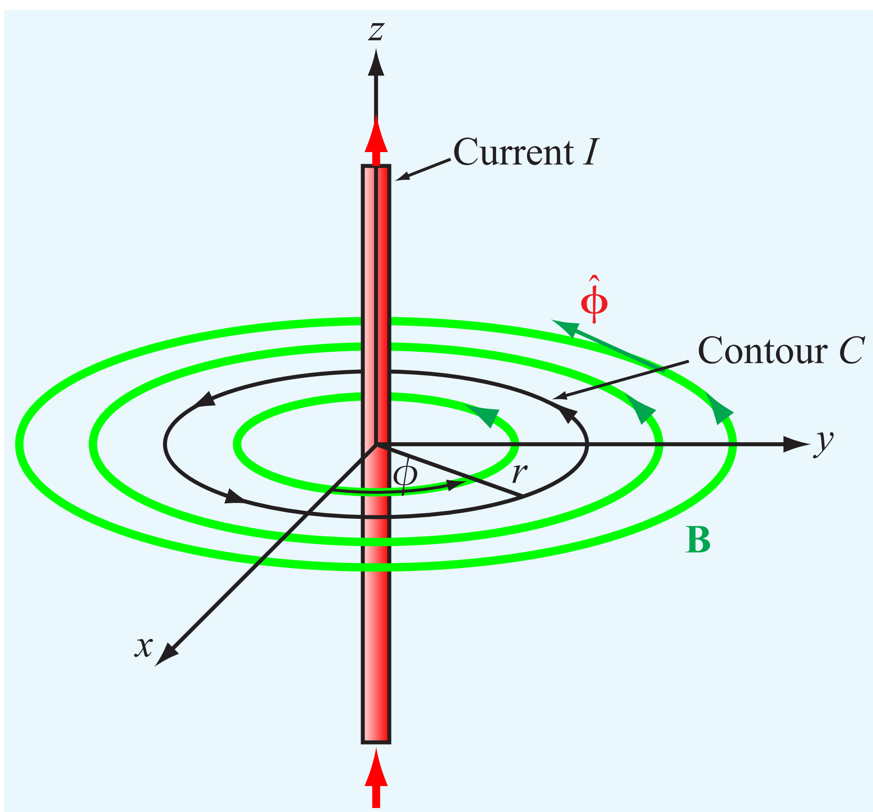

Figure 3.25: An azimuthal field, $\mathbf{B} = \hat{\boldsymbol{\phi}} \mu_0 I/(2\pi r)$ with circulation around the z-axis.

- **Finally** we cab defined curl as

$$\nabla \times \mathbf{B} = \text{curl}\,\mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l}\right]_{\max}$$

- **Note**: The contour C is oriented to given the maximum circulation; position the paddle wheel so it spins the fastest

- Since $\nabla \times \mathbf{B}$ is a vector, its direction is $\hat{\mathbf{n}}$, the unit normal of surface Δs (use the right-hand rule with the fingers curling in the direction of C and the thumb pointing along $\hat{\mathbf{n}}$)

- In rectangular coordinates we compute the curl via

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

- For other coordinate systems consult the back page of the text

Stoke's Theorem

- *Stoke's theorem* converts a surface integral of the curl to a line integral of a vector along a contour C bounding surface S

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

3.7 Laplacian Operator

- The *Laplacian operator* shows up in a number of contexts
- The text mentions the divergence of the gradiant, $(\nabla \cdot (\nabla V))$ as one possibility
- The result is known as *del square*

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

- In Chapter 4 Laplace's equation, $\nabla^2 V = 0$, arises when determining the electrostatic potential in 1D, 2D, and 3D problems